A Study of Lorenz Links

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Abstract

In this thesis we study the class of Lorenz links, which arises from the periodic orbits of the Lorenz system. This class of links has been studied first in 1980s by J. Birman, and R. F. Williams using a geometric model called the Lorenz template. It turns out that Lorenz links have very rich properties. There are also connections with many areas in Mathematics. For instance, in 2006 E. Ghys has proved that the class of Lorenz links coincide with the class of modular links, which can be seen as the periodic orbits of the geodesic flow on the modular surface. We will demonstrate some of the properties of the Lorenz links and also give an alternative way to establish the correspondence of Lorenz links with T-links, which is recently proved by J. Birman and I. Kofman in 2009.

摘要

本論文將研究源自Lorenz 系統的周期軌線的一類鏈環, 此類鏈環被稱作Lorenz 鏈環。於1980年代J. Birman 及 R. F. Williams 他們利用Lorenz系統的模型對Lorenz 鏈環作出研究。Lorenz 鏈環擁有非常豐富的性質，與數學的不同領域也有聯繫。E. Ghys 證明了模鏈環(可以看作Modular surface上測地流的周期軌線)與Lorenz 鏈環之間的對應。本文我們將探討Lorenz鏈環的其中一些性質。J.Birman 與I. Kofman 證明了Lorenz鏈環與T-鏈環之間對應，我們將給出另一個證明方法。

ACKNOWLEDGMENTS

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Contents

Chapter 1

Introduction

A periodic orbit of a system of differential equations in $\mathbb{R}^3$ defines a knot and a finite collection of periodic orbits hence defines naturally a link. In this paper we study the class of links which arises from the Lorenz's equations (The original parameters are $\sigma = 10, b = \frac{8}{3}$ and $r = 28$):

$$\begin{cases} \frac{dx}{dt} = \sigma(y - x) \\ \frac{dy}{dt} = rx - y - xz \\ \frac{dz}{dt} = xy - bz \end{cases}$$

This system is originally introduced by E. Lorenz in 1963 as a model of atmospheric convection and is famous for its chaotic behaviour. Fig.1.1 shows a typical orbit of the system. Observe that the orbits seem to wind around the two equilibrium points and eventually tend to a set which looks like a branched surface, the so-called Lorenz attractor.

In order to study the orbits, J. Guckenheimer and R. Williams [10] introduced a model, the Lorenz template(or called the geometric Lorenz attractor), to describe the behaviour of the orbits.It was verified by Tucker [20] that the model does corresponds to the Lorenz system for certain parameters.

The Lorenz template(Fig.1.2) is a branched surface in $\mathbb{R}^3$ with a semi-flow(as indicated by the arrows). Each periodic orbit either goes left or right around the corresponding "hole" each time it reaches the branch line, which is denoted by I. In [4] J. Birman and R. Williams studied the periodic orbits using the Lorenz template. Each periodic orbit is given a code that describes it. We will give a description of this in Chapter 2.

The Lorenz template naturally determines a positive braid representation for the Lorenz links. As a result, Lorenz links inherit many properties from the positive braids. In Chapter 3 we will present some properties of Lorenz links as closed positive braids. Such properties are known at the time of [4].

J. Birman and I. Kofman proved in [3] that Lorenz links actually coincides with another class of links represented by positive braids called the T-links. T-links are generalization of torus links by repeated positive twisting. In Chapter 4 we will prove this correspondence using the algebraic relations of the braid group, and give an exposition to a result about the braid index of Lorenz links[11].

In the last chapter we give a description of the correspondence of Lorenz links and Modular links, which is proved by E. Ghys in [8].

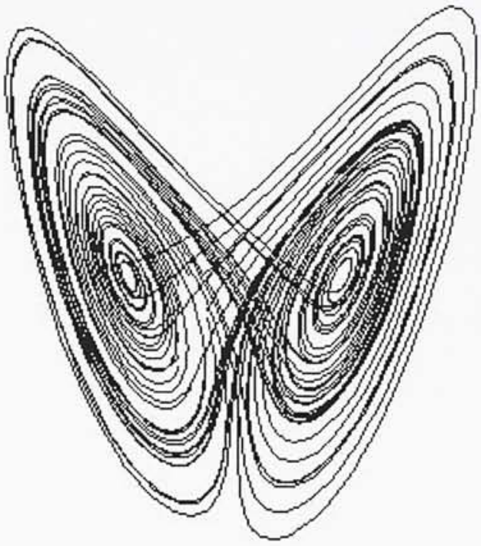

Figure 1.1:

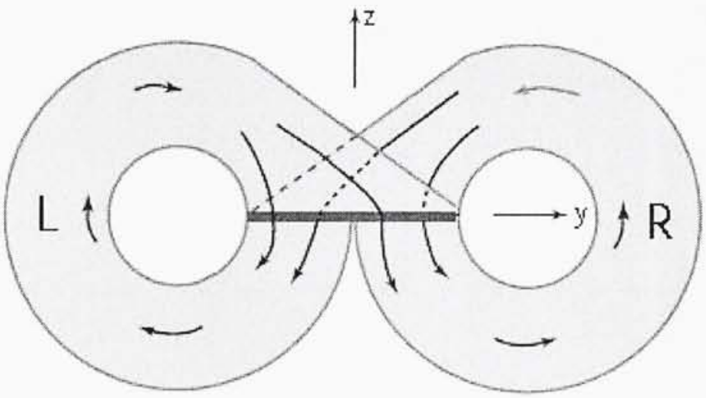

Figure 1.2: The Lorenz template[3]

Chapter 2

Coding the Lorenz knots

In [4] J. Birman and R. Williams studied the symbolic dynamics on the Lorenz template, which gives a way to code the Lorenz knots. In this chapter we are going to present their results.

2.1 Lorenz words

Recall the Lorenz template introduced in the previous chapter. The semi-flow on the Lorenz template is described by the first return map f on the branch line I.

Here we briefly describe this model for the Lorenz system. The Lorenz system (with the original parameters) have 3 equilibrium points, one at the origin and two at the plane $z = 27$. As the trajectories keep revolving around the latter two points, a first return map is constructed on a portion of the plane in which those two points lie. By linearizing the system one sees that there are one unstable direction and two stable directions, one of which is vertical. In the model it is assumed that the system behaves like a linear system near the origin and then goes around the equilibrium points. It is assumed that there is a strong stable foliation parallel to the stable direction near the origin. Collapsing the foliation results in the branched surface. See [4], [10] and further references given there for a more detailed description.

The branch line I is identified with the unit interval [0,1], and the first returned map $f : I\backslash\{\frac{1}{2}\} \to I$ is taken to be a map with $f([0, \frac{1}{2})) = [0, 1)$, $f((\frac{1}{2}, 1]) = (0, 1]$ and $f' > \lambda > 1$. (f is not defined at $\frac{1}{2}$ to avoid ambiguity.)

For each point $x \in I\backslash\{\frac{1}{2}\}$, its orbit is described by the sequence $x, f(x), f^2(x), \ldots$. The orbit of x to $f(x)$ goes around the left hole if $x < \frac{1}{2}$ and goes round the right hole if $x > \frac{1}{2}$. If we consider the set of points which never meet $\frac{1}{2}$ by successive applications of f, to each point we can associate a sequence $a_0a_1a_2\ldots$ in the symbols L or R such that a_i represents the relative position of the point to the midpoint after the i-th application of f, with L and R meaning to the left and right respectively.

It turns out that[4] we can take f to be the doubling map $t \mapsto 2t(mod\mathbb{Z})$. Hence for the following discussion we take f to be this map.

Note that if we express a point in $I\backslash\{\frac{1}{2}\}$ in its binary representation, the application of f corresponds to a left-shift:

For $x = 0.a_0a_1a_2\ldots$, $a_i \in \{0,1\}$, $f(x) = 0.a_1a_2a_3\ldots$.

Hence x is a periodic point if and only if its binary representation is periodic. Note that the points 0 and 1 are fixed points of f and these correspond to 2 trivial knots encircling the two holes at each side of the interval I. We call a periodic orbit of the semi-flow on the Lorenz template which does not correspond to the two trivial knots above a *Lorenz knot.* A *Lorenz link* is a finite collection of Lorenz knots.

Observe that for a periodic point $x = 0.a_0a_1a_2\ldots$, a_i indicates the relative position of x to the mid-point $\frac{1}{2}$ of I: 0, 1 coresponds to the left, right of $\frac{1}{2}$ respectively. Therefore, the sequence in L, R described above is obtained by simply taking the sequence after the decimal point, and changing 0, 1 to L, R respectively. A periodic orbit is then uniquely determined by a periodic sequence of symbols L and R.

Theorem 2.1. *The class of periodic orbits corresponds one to one with the cyclic permutation class of finite aperiodic words in L and R of length greater than of equal to 2.*

Proof. Given a periodic point $x = 0.a_0a_1 \ldots a_{n-1}$ with minimal period n, it is identified with an aperiodic word w of length n obtained by changing 0, 1 in $0.a_0a_1 \ldots a_{n-1}$ to L, R respectively. Each cyclic permutation of such a word gives a point in the orbit of x, and the assumption that it is aperiodic corresponds to the fact that minimal period is used in the identification. The length is assumed to be at least 2, so the two trivial orbits 0 and 1 are omitted. $\square$

Definition 2.1. *A **Lorenz word** is a word described in Theorem 2.1.*

Given a Lorenz word w, we can identify an orbit and as well as a unique point in I. This gives a way for recovering the corresponding Lorenz link of a given set of Lorenz words. We describe this algorithm below:

(1) Given Lorenz words $w_1, \ldots, w_n$, list all the possible cyclic permutations.

(2) Arrange the words in ascending order with respect to the lexicographical order.

(3) Plot the points on the branch line in this order and to the correct side of the midpoint.

(4) Finally complete the orbits along the flow by joining a point to the point that comes next in the cyclic permutation.

For example, the word $LLRLR$ is the Lorenz knot in Fig.2.1

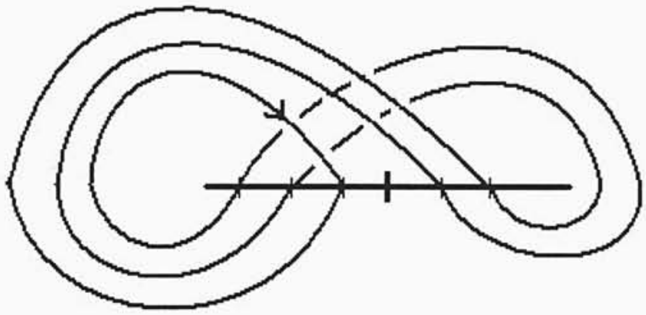

Figure 2.1: The knot LLRLR

Chapter 3

Lorenz links and positive braids

Lorenz links have a natural representation by positive braids, and there are some properties that can be deduced from this. The arguments are from [4], [5], [6] and [7].

3.1 Lorenz braids

In the previous chapter, the definition of a Lorenz link exclude the possibility of parallel copies of the same orbit, since given a word describing left and right at each step, exactly one orbit on the Lorenz template is identified. However, in the following discussion, we would like to include such possibilities as Lorenz links.

To be precise of what links are considered as Lorenz links, we proceed as follows:

By cutting open the Lorenz template along the left and right band, each periodic orbit is naturally identified with a braid with an induced orientation. Taking closure of this braid the original link is obtained.

Such a braid consists of a group of strands on the left going from left to right and the remaining strands on the right go from right to left. The strands belonging to the first group are on the top and those in the other group are at the bottom, and the strands in the same group do not cross each other. Such a braid is completely determined by the permutation of the strands in the first group.

Consider a sequence of integers $1 \leq d_1 \leq \ldots \leq d_p$ for $p > 1$. A vector $d_L = (d_1, \ldots, d_p)$ is called a *Lorenz vector*. This vector defines a unique permutation π of $1, \ldots, p + d_p$ which satisfies

$$1 = \pi(p+1) < \ldots < \pi(p+d_p) \text{ and } \pi(1) < \ldots < \pi(p) = n.$$

Such permutation is called a *Lorenz permutation*. It determine a braid of $n = p+d_p$ strands by joining i to $\pi(i)$ with straight lines and the first p strands are on the top. We call a braid obtained from this process a *Lorenz braid*. An example showing the braid determined by the vector (2,2,3) is shown in Fig.3.1.

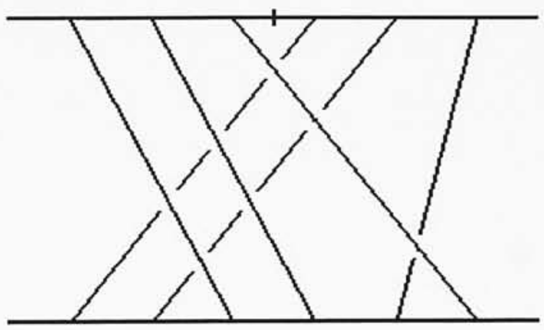

Figure 3.1: The Lorenz braid corresponding to (2,2,3)

From now on we use the following definition for Lorenz links:

Definition 3.1. *A **Lorenz link** is a link obtained by the closure of a Lorenz braid.*

In other words, we consider all possible patterns which can be drawn on the Lorenz template.

3.2 Properties of Lorenz links as the closure of a positive braid

A Lorenz braid has an orientation naturally induced from the flow on the Lorenz template, and all the crossing are of the same type. We call this type of crossing to be positive, and the other type negative. Note that this is opposite to the usual convention in knot theory.

Hence by our convention Lorenz braids and all positive braids, and we can deduce some properties from this.

The following can be seen easily by drawing a few pictures:

Theorem 3.1. *[4] Any two components in a Lorenz link have positive linking number. In particular, any Lorenz link is unsplittable.*

Proof. Let K_1 and K_2 be two components of a Lorenz link. Consider the corresponding Lorenz braid representation. Since all crossing are positive, it suffices to show there is a crossing between the two Lorenz knots. This holds true because each Lorenz knot has at least one overcrossing strand and one undercrossing strand. $\square$

The next result is about the genus of a Lorenz link.

Recall that a *Seifert surface* of an oriented link is an orientable surface with the link as the oriented boundary. Given a diagram of an oriented link, a Seifert surface can be constructed by the Seifert algorithm: By smoothing the crossings according to the orientation the diagram becomes disjoint simple closed curves called *Seifert circles*. By forming a disc for each Seifert circle and joining them with half-twisted bands for each crossing we obtain a Seifert surface. We call such a surface a *projection surface.*

Given a projection surface, we can construct a graph with one vertex for each Seifert circle and connect two vertices with an edge for each half twisted band connecting their corresponding Seifert circles. We call such a graph a *Seifert graph.* The *rank* of the Seifert graph is the number of edges minus the number of edges in a spanning tree.

There is a canonical way to form a projection surface of a braid. The projection surface of a positive braid is actually of minimal genus.

To present the proof, we introduce an invariant of oriented links called the Homfly polynomial. There are different conventions and we follow [5], [6]. The arguments concerning the Homfly polynomial are also from there.

Definition 3.2. *The Homfly polynomial of an oriented link L, denoted by $P(L)$ or $P_L(v,z)$, is defined by the following three axioms.*

1. $P_L(v,z)$ is invariant under ambient isotopy of L.

2. For the trivial knot K, $P_K(v,z) = 1$.

3. If L_+, L_- and L_0 are three oriented links having diagrams D_+, D_- and D_0 which differ in a neighborhood as shown in Fig.3.2, then the corresponding Homfly polynomial satisfies $v^{-1}P(L+) - vP(L_-) = zP(L_0)$.

Figure 3.2: The skein relation

It is well-known that the Homfly polynomial is a well-defined invariant of oriented links and takes values in $\mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$.

Axiom 3 in the definition is called a *Skein relation*).

The Homfly polynomial is a generalization of other link invariants. In particular, $\Delta_L(x) = P_L(1, x^{-1} - x)$ and $\nabla_L(z) = P_L(1, z)$ are the Alexander polynomial and the Conway polynomial respectively.

To compute the Homfly polynomial of a link L, we make use of the skein relation and repeatedly convert the link to other simpler links. It can be shown that the homfly polynomial a trivial link of μ components is $\delta^{\mu-1}$, where $\delta = \frac{v^{-1}-v}{z}$.

To carry out the computation process we make use of resolution trees. A resolution tree is a binary tree such that each node represents a diagram and each edge represents a monomial. The root represents a diagram of L and each non-terminal node and its two children are related as shown in Fig.3.3.

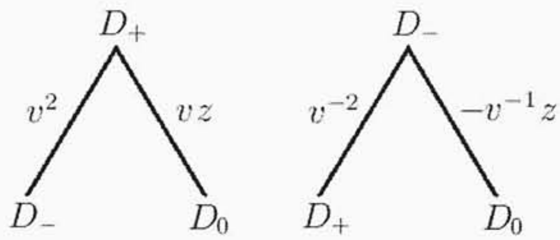

Figure 3.3:

Let Γ be the set of terminal nodes. For each $\gamma \in \Gamma$, let $m_\gamma(v,z)$ be the product of monomials along the path from the root to γ and $P_\gamma(v,z)$ be the Homfly polynomial of the diagram at γ. Then the Homfly polynomial of the original link L is given by $\sum_{\gamma\in\Gamma} m_\gamma P_\gamma$. The goal is to construct a resolution tree with all terminal nodes representing trivial links.

This is proved to be possible. Indeed, this can be done in a systematic way: it is possible to construct a resolution tree such that no crossing is changed more than once along any path [5], [6]. The idea is to convert the diagram along any path into an ascending diagram (hence it represents a trivial link) by appropriate choice of basepoints and ordering of the components at each step. It is also possible to carry out resolution such that the Seifert graphs remain connected at each node.

Theorem 3.2. *Let D be a connected, positive diagram of a link L. Let F be the projection surface constructed from D and let G be its Seifert graph. Then $\nabla_L(z)$ has a term of degree $rank(G)$ and is the term of the highest degree.*

Proof. The proof is by induction on the number of crossing $c(D)$ of D.

When $c(D) = 0$, the link is a trivial knot and the Seifert graph consists of a single vertex. $rank(G) = 0$ and $\nabla_L(z) = 1$.

When $c(D) > 0$, start a resolution by choosing appropriate basepoints and ordering of components, and require that the Seifert graphs to remain connected. Since $\nabla_L(z) = P_L(1, z)$, the monomial on the left edge is always 1. The monomial on the right edge is always z as all crossings are positive and no crossing is changed more than once.

Choosing the path on the right at a node corresponds to smoothing a crossing and deleting the edge representing that crossing in the Seifert graph. At most $rank(G)$ edges can be deleted for G to remain connected. Therefore the degree of z is at most $rank(G)$.

We need to show that such a term exists. At the rightmost terminal node the diagram D' has fewer crossings than D. Suppose it takes p steps to reach this node. Then Seifert graph G' of D' has rank $rank(G) - p$. By the induction hypothesis $P(D')$ has a term of degree $rank(G) - p$. However, $\nabla(D')$ is 1 when D' represents a trivial knot and is 0 when D' represents a trivial link with at least two components. Hence $p = rank(G)$ and G' is a tree, which represents a trivial knot.

Therefore, along the rightmost path we obtain a term of degree $rank(G)$, which does not cancel with other terms as all terms are positive in the resolution tree.

□

To finish the proof that the projection surface is of minimal genus, recall that the Alexander polynomial $\Delta_L(x)$ is also obtained by $det(xM - x^{-1}M^T)$, where M is a Seifert matrix of L. If M is constructed using a Seifert surface F', then M is a $(2g(F') + \mu(L) - 1) \times (2g(F') + \mu(L) - 1)$ matrix. (g denotes the genus and μ denotes the number of components. And we remark that we follow the convention in [6] for the definition of the Alexander polynomial.)

Therefore,

$$2maxdeg_z(\nabla(L)) = breadth_x(\Delta(L)) \leq 2g(F') + \mu(L) - 1.$$

From Theorem 3.2, the lower bound is achieved by the projection surface F, as $rank(G) = 2g(F) + \mu(L) - 1$.

As a result,

Theorem 3.3. *[4] Let L be a Lorenz link of μ components. Suppose the corresponding Lorenz braid has n strands and c crossings. Then*

$$2g(L) = c - n - \mu + 2.$$

Proof. By triangulating each Seifert circle into $2c$ triangles and each half twist band into 2 triangles. We have

$$\chi(L) = (2cn + n) - (4cn + 3c) + (2cn + 2c) = n - c.$$

The result follows as $1 - \chi(L) = 2g(L) + \mu - 1$. □

We remark that the result of Theorem 3.2 holds for a larger class of links called *homogeneous links*, the version above is a simplified one for positive links. See [5] and [6] for more details.

Another property of Lorenz links is that they are fibered. Recall that a link L is *fibered* if the its complement $\mathbb{S}^3\ L$ in $\mathbb{S}^3$ fibers over $\mathbb{S}^1$, with fiber a Seifert surface of L.

Here we follow the argument given in [7].

The link shown in Fig.3.4 is called the Hopf link. It can be defined as the set H of $(z_1, z_2) \in \mathbb{C}^2$ satisfying $P(z_1, z_2) = (z_1 - \lambda_1 z_2)(z_1 - \lambda_2 z_2) = 0$ and $|z_1|^2 + |z_2|^2 = 1$, where λ_1 and λ_2 are two distinct constants in $\mathbb{C}$. The function $f : H \to \mathbb{S}^1$ defined by $f(z_1, z_2) = \frac{P(z_1,z_2)}{|P(z_1,z_2)|}$ defines a fibration of the Hopf link.

Figure 3.4: The Hopf link

The idea of the proof is that the closure of a positive braid can be constructed by "adding" Hopf links together, and at each step the link obtained is fibered. This is done by the *Murasugi sum*.

Definition 3.3. *Let Σ_1, Σ_2 be two surfaces embedded in $\mathbb{R}^3$ with boundary L_1 and L_2 respectively. Let Π be a plane separating $\mathbb{R}^3$ into two open balls B_1 and B_2. Suppose that*

(1) $\Sigma_1 \subseteq \bar{B}_1$ and $\Sigma_2 \subseteq \bar{B}_2$;

(2) $\Sigma_1 \cap \Sigma_2$ is a $2n$-gon D contained in Π;

(3) L_1, L_2 are two links intersecting at the vertices $x_1, \ldots, x_{2n}$ of D.

The ***Murasugi sum*** $\Sigma_1 \sharp \Sigma_2$ *of* Σ_1 *and* Σ_2 *is defined as their union* $\Sigma_1 \cup \Sigma_2$*.*

The Murasugi sum $L_1 \sharp L_2$ *of* L_1 *and* L_2 *is the link* $L_1 \cup L_2 \setminus \bigcup_i (x_i, x_{i+1})$*.*

Fig.3.5 shows an example of Murasugi sum.

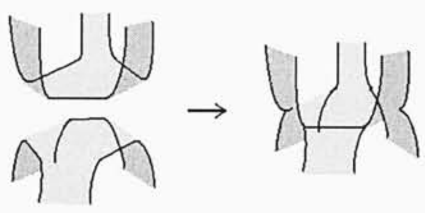

Figure 3.5: Murasugi sum

The Murasugi sum preserves the property of links of being fibered.

Theorem 3.4. *If* L_1*,* L_2 *are two fibred links, then their murasugi sum* $L_1 \sharp L_2$ *is also a fibred link.*

The idea of the proof is that the fibers of the two links can be arranged to meet nicely and this allows one to define the fibers for the murasugi sum. The reader is referred to [7] for the argument.

By the above fact,

Theorem 3.5. *A link which can be obtained by the closure of a connected positive braid is fibered. In particular, any Lorenz link is fibered.*

Proof. The goal is to construct the link by forming appropriate Murasugi sums of the Hopf link or the trivial knot.

A 2-strand braid with $n > 1$ positive crossings, $n \geq 2$, can be constructed by forming Murasugi sums of Hopf links $(n-1)$ times. The case $n = 1$ corresponds to a half-twisted band and represents a trivial knot. Then the 2-strand braids are put together by appropriate Murasugi sums to form a braid with more strands, which corresponds to the required link. For a picture illustrating how the Murasugi sums should be formed, the reader is referred to [7]. □

Chapter 4

T-Links and the braid index

In the following chapter we will discuss the correspondence of Lorenz links and T-links and related results, which are proved in [3]. A discussion on the braid index of Lorenz links is also included in this chapter.

We first introduce some notations.

The braids of n strands form a group $\mathbb{B}_n$. It is generated by $(n-1)$ generators $\sigma_1, \ldots, \sigma_{n-1}$, where σ_i represents the braid shown in Fig.4.1.

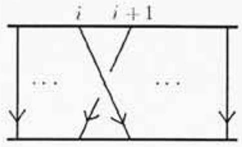

Figure 4.1:

These generators satisfy the following relation:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| \geq 2$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

There are also moves called *Markov moves* which keep the closure of the braid invariant:

(1) Conjugate $\beta \in \mathbb{B}_n$ to $\beta' \in \mathbb{B}_n$.

(2) Replace $\beta \in \mathbb{B}_n$ by $\beta {\sigma_n}^{\pm 1} \in \mathbb{B}_{n+1}$ and vice versa.

(3) Replace $\beta \in \mathbb{B}_n$ by $\beta {\sigma_1}^{\pm 1} \in \mathbb{B}_{n+1}$ and vice versa.

In view of (2) and (3), we restrict our discussion to the Lorenz links with Lorenz vectors $(d_1, \ldots, d_p)$ satisfying $d_1 \geq 2$ and $d_{p-1} = d_p$. It is also convenient to write a Lorenz vector in a reduced form: If the ordered p-tuple $(d_1, \ldots, d_p) = (r_1, \ldots, r_1, r_2, \ldots, r_2, \ldots, r_k, \ldots, r_k)$ and each r_i appears s_i times in the sequence, then we write it as $({r_1}^{s_1}, \ldots, {r_k}^{s_k})$. For example, $(2,2,3,5,5,5,6,6)$ in reduced form is $(2^2, 3, 5^3, 6^2)$.

4.1 T-links

Definition 4.1. *A T-link is a link represented by a braid of the form*

$$(\sigma_1 \dots \sigma_{r_1-1})^{s_1}(\sigma_2 \dots \sigma_{r_2-1})^{s_2} \dots (\sigma_1 \dots \sigma_{r_k-1})^{s_k},$$

where $2 \leq r_1 < r_2 < \dots < r_k$ *are integers.*

It is denoted by $T((r_1, s_1) \dots (r_k, s_k))$.

Figure 4.2 shows the braid corresponding to $T(3,2)(4,1)$. Note that this generalize the usual torus links $T(p,q)$.

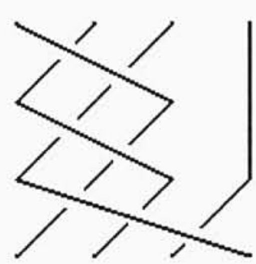

Figure 4.2: $T((3,2)(4,1))$

In the following we are going to show that the Lorenz link $({r_1}^{s_1}, \dots, {r_k}^{s_k})$ and the T-link $T((r_1, s_1) \dots (r_k, s_k))$ are isotopic to each other. This is first proved in [3] by Birman and Kofman. Their proof is by deforming the closed braid and uncoiling certain strands in the link. Here we give another proof using algebraic relations of the braid words, which makes the uncoiling more transparent.

To simplify notations, we define $\mathbb{B}_{m,n}$ to be the braid group of braids with $n-m+1$ strands. The strands are labeled from m to n instead of from 1 to $n-m+1$. The generators are then $\sigma_m, \ldots, \sigma_{n-1}$, which correspond to $\sigma_1, \ldots, \sigma_{n-m+1} \in \mathbb{B}_{n-m+1}$ in the natural way.

The following property [3] of the generators of the braid group would also be useful. Let $[m, n]$ denote the product $\sigma_m \ldots \sigma_n$.

Lemma 4.1. *Let $m \le p \le q < n$. Then $[m, n][p, q] = [p + 1, q + 1][m, n]$.*

Proof. It suffices to show that $[m, n]\sigma_i = \sigma_i + 1[m, n]$ for $m \le i < n$.

By the relations of the generators,

$$\begin{aligned} [m, n]\sigma_i &= [m, i-1]\sigma_i\sigma_{i+1}\sigma_i[i + 2, n] \\ &= [m, i-1]\sigma_{i+1}\sigma_i\sigma_{i+1}[i + 2, n] \\ &= \sigma_{i+1}[m, n]. \end{aligned}$$

□

Theorem 4.1. *The Lorenz link $({r_1}^{s_1}, \ldots, {r_k}^{s_k})$ and the T-link $T((r_1, s_1) \ldots (r_k, s_k))$ are isotopic.*

Proof. The theorem is proved by performing successive "untwisting and sliding" of the crossings in the Lorenz braid determined by $({r_1}^{s_1}, \ldots, {r_k}^{s_k})$. We prove that we can transform the Lorenz braid into the required braid of the T-link.

It is easier to prove the theorem using the original Lorenz vector expression. Hence our goal is to transform the Lorenz braid into $\prod_{i=1}^{p}[1, d_i - 1] \in \mathbb{B}_{d_p}$

Consider the Lorenz vector $(d_1, \ldots, d_p)$. In terms of the generators in $\mathbb{B}_{1,n}$, $n = p + q$, $q = d_p$, the corresponding Lorenz braid is

$$(\sigma_p \ldots \sigma_{p+d_p-1})(\sigma_{p-1} \ldots \sigma_{(p-1)+d_{p-1}-1}) \ldots (\sigma_1 \ldots \sigma_{d_1}).$$

We write it as

$$\begin{array}{cccccc} & & & \sigma_p & \cdots & \sigma_{p+d_p-1} \\ & & \sigma_{p-1} & \sigma_p & \cdots & \sigma_{(p-1)+d_{p-1}-1} \\ & & \vdots & & & \\ \sigma_1 & \cdots & \sigma_{d_1} & & & \end{array} \quad (\in \mathbb{B}_{1,n}),$$

which resembles better the picture of the braid.

For example, $(2^2, 3, 5^2)$ is expressed as

$$\begin{array}{ccccccccc}
 & & & & \sigma_5 & \sigma_6 & \sigma_7 & \sigma_8 & \sigma_9 \\
 & & & \sigma_4 & \sigma_5 & \sigma_6 & \sigma_7 & \sigma_8 & \\
 & & \sigma_3 & \sigma_4 & \sigma_5 & & & & \\
 & \sigma_2 & \sigma_3 & & & & & & \\
\sigma_1 & \sigma_2 & & & & & & &
\end{array}.$$

As σ_1 appears only once, we can apply a Markov move to "untwist" the braid and obtain a braid in $\mathbb{B}_{2,n}$:

$$\begin{array}{ccccccc}
 & & & \sigma_p & \cdots & \sigma_{p+d_p-1} & \\
 & & \sigma_{p-1} & \sigma_p & \cdots & \sigma_{(p-1)+d_{p-1}-1} & \\
 & & \vdots & & & & (\in \mathbb{B}_{2,n}). \\
\sigma_2 & \cdots & \sigma_{2+d_2-1} & & & & \\
\sigma_2 & \cdots & \sigma_{d_1} & & & &
\end{array}$$

Then we "push" the last row to the top by applying $(p-1)$ times Lemma 4.1. In the picture, this pushes the strand starting on the leftmost position to the top.

$$\begin{array}{ccccc}
[p, p+d_p-1] & & [p, p+d_p-1] & & [\mathbf{p+1}, \mathbf{p+d_1-1}] \\
[p-1, p+d_{p-1}-2] & & [p-1, p+d_{p-1}-3] & & [p, p+d_p-1] \\
\vdots & = & \vdots & = \ldots = & [p-1, p+d_{p-1}-2] \\
[2, 1+d_2] & & [\mathbf{3}, \mathbf{d_1+1}] & & \vdots \\
[\mathbf{2}, \mathbf{d_1}] & & [2, 1+d_2] & & [2, 1+d_2]
\end{array}.$$

The process is valid as the row being pushed up always begin with the same generator as the row just on top of it, but is shorter in length (as $d_1 \leq \ldots \leq d_p$).

Now σ_2 appears only once in the expression, we may remove it and get a braid in $\mathbb{B}_{3,n}$. Then we can apply Lemma 4.1 again to push the bottom row up until it starts with σ_{p+1}.

In a total of $(p-1)$ steps of "untwisting and pushing", we finally arrive at

$$\begin{matrix} [p+1, p+d_1-1] \\ [p+1, p+d_2-1] \\ \vdots \\ [p+1, p+d_p-1] \end{matrix} \in \mathbb{B}_{p+1,n},$$

which is the same as the required braid $\prod_{i=1}^{p}[1, d_i - 1] \in \mathbb{B}_{d_p}$ □

Fig.4.3 illustrates the process of untwisting and pushing.

To describe it in words, a Lorenz link is transformed to a T-link by repeating the process of untwisting the left most crossing and then pushing the leftmost strand under the other strands to the top.

By Theorem 4.1,

Theorem 4.2. *There is a one to one correspondence between the isotopy classes of Lorenz links and T-links.*

Therefore, the properties of Lorenz links are also satisfied by T-links.

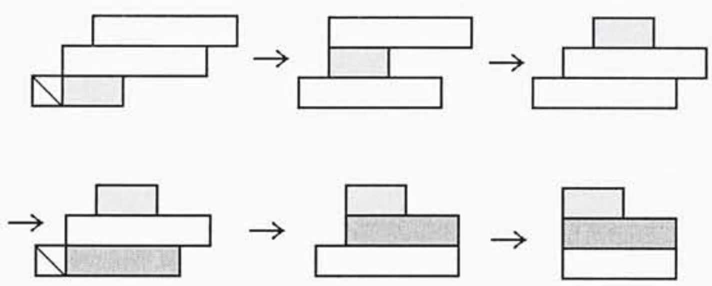

Figure 4.3:

In [18], Razumovsky gives another method to visualize the correspondence between Lorenz link to T-link. He shows that both the Lorenz link and T-link can be obtained by deforming a link determined by the permutation corresponding to the Lorenz link. Please see [18] for the details.

4.2 Symmetries

Each braid in $\mathbb{B}_n$ has a natural symmetry by rotating the braid by 180 degrees along a vertical axis. In terms of Markov moves, this is done by conjugating the whole braid by the half twist $\Delta_n = \prod_{i=1}^{n-1}[1, n-i]$, which has the effect of sending σ_i to σ_{n-i}. It has For a Lorenz braid L, rotating the braid by 180 results in another Lorenz braid, which is called the *dual Lorenz braid* and is denoted by $\overline{L}$.

Let $d_L = ({r_1}^{s_1}, \ldots, {r_k}^{s_k})$ be the Lorenz vector defining L. It is easy to verify that the dual Lorenz vector $d_{\overline{L}}$ is of the form $({u_1}^{v_1}, \ldots, {u_k}^{v_k})$.

If we set $r_0 = 0$, then

$$u_i = \sum_{j=1}^{i} s_{k-j+1} \text{ and } v_i = r_{k-i+1} - r_{k-i} \tag{4.1}$$

Using the correspondence in Theorem 4.1, we can identify the corresponding T-Links.

Theorem 4.3. *The T-links $T((r_1, s_1) \ldots (r_k, s_k))$ and $T((u_1, v_1) \ldots (u_k, v_k))$, where u_i, v_i are given by (4.1), have the same link type.*

This generalizes the correspondence between the torus links $T(p, q)$ and $T(q, p)$, which is as a special case of Theorem 4.3.

T-links with certain representations represent the same link, which allow us to draw the same conclusion on Lorenz links. This is proved by Birman and Kofman in [3]. We are going to prove it using the same idea as in proving Theorem 4.1.

Theorem 4.4. *Let L_1 be the Lorenz braid defined by $d_L = ({r_1}^{s_1}, \ldots, {r_k}^{s_k})$ such that for $i = 1, \ldots, k-1$, $s_i = n_i r_i$ for some positive integers n_i and $r_{k-1} \leq s_k$. Let L_2 be the Lorenz braid defined by $d_L = ({r_1}^{s_1}, \ldots, {r_{k-1}}^{s_{k-1}}, {s_k}^{r_k})$. Then L_1 and L_2 represents the same link.*

Proof. From the proof of Theorem 4.1, $T((r_1, s_1) \ldots (r_k, s_k))$ has a braid representation as shown on the left of Fig.4.4.

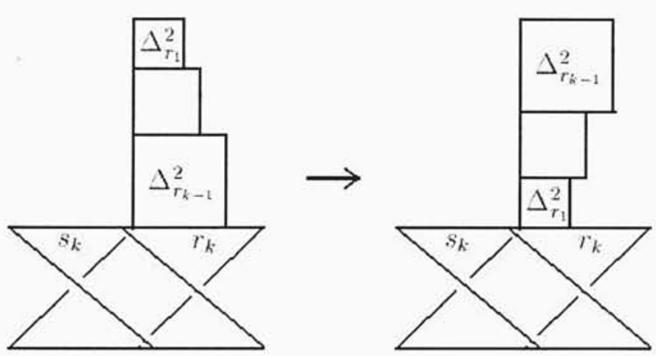

Figure 4.4:

The blocks on the top are full twists due to the assumption that $r_i | s_i$ for $i = 1, \ldots, k-1$. The full twists $\Delta^2_{r_i}$ are all starting from the $s_{k+1} - th$ strand from the left and for simplicity all n_i are taken to be 1.

Since a full-twist of l strands commute with any braid in those l strands, we can rearrange the full twists as on the right of Fig.4.4

Push all the full-twists to the bottom by applying Lemma 4.1 repeatedly, and rotate the braid by 180 degrees horizontally. (Fig.4.5)

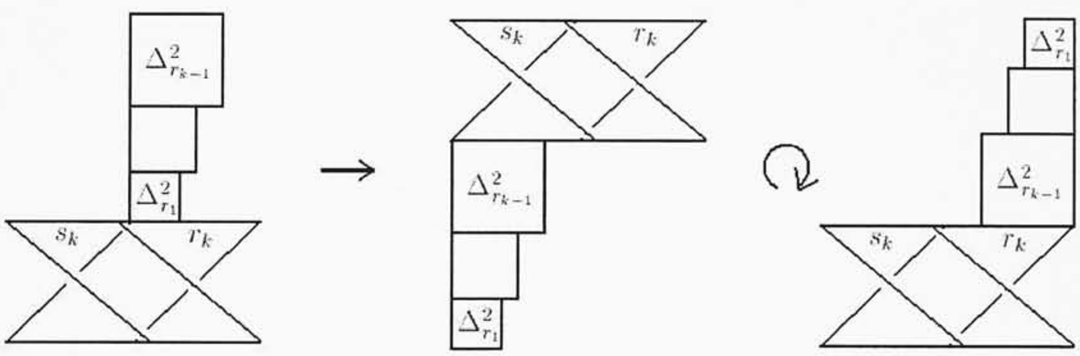

Figure 4.5:

Then conjugate the full-twists to the top and rotate the braid along a vertical axis. (Fig.4.6) Note that the full-twists still appear as full-twists.

Finally we push the full-twists up, and the resulting braid is seen to be represent the T-link $T((r_1, s_1)(r_2, s_2)\dots(s_k, r_k))$(Fig.4.6).

□

We remark here the orientation is reversed when we rotate the braid horizontally. Hence if we give the links orientation induced by the Lorenz template, then L_1

corresponds to L_2 with the opposite orientation.

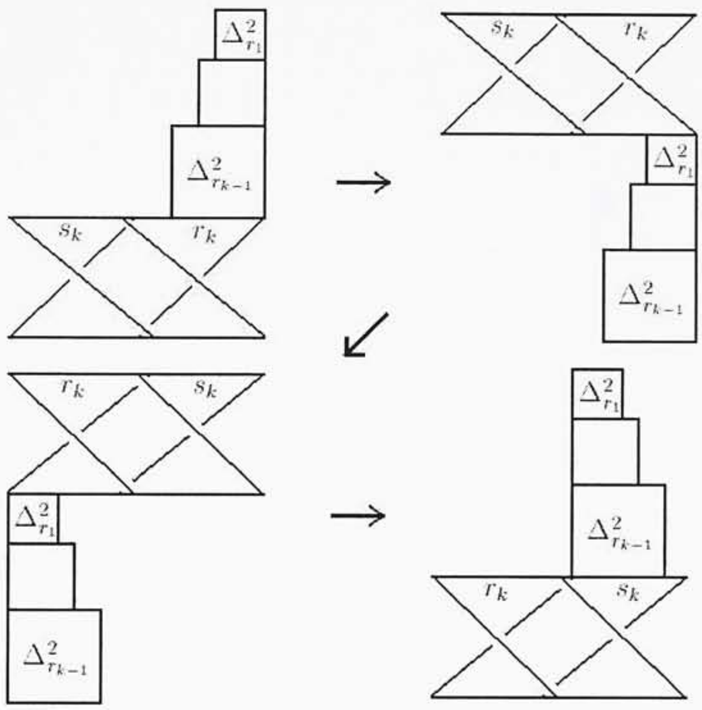

Figure 4.6:

4.3 Trip number and the braid index

Consider a Lorenz braid defined by $(d_1, \ldots, d_p)$. The strands of a Lorenz braid can be divided into 4 groups: LL, LR, RL and RR. A strand in the LR group starts on the left and ends on the right. The others are defined similarly.

Definition 4.2. *The **trip number** t of a Lorenz link is the number of LR strands.*

If π is the permutation associate to the Lorenz braid, then

$$t = \text{number of } i \in \{1, \dots, p\} \text{ satisfying } \pi(i) > p.$$

Also note that the number of RL stands is also equal to t, so the dual Lorenz braid has the same trip number.

Lemma 4.2. *$d_i \leq t$ for $i = 1, \dots, p-t$*

Proof. Consider the last LL strand, which is the $(p-t)$-th strand. Then

$$(p-t) + d_{p-t} \leq p, \text{ so } d_{p-t} \leq p$$
$$\text{and } d_1 \leq \dots \leq d_{p-t} \leq p.$$

□

Observe that the t RL strands enters the left of the Lorenz template, go around the left hole for different number of times and eventually becomes the t LR strands. It is natural to ask if there is a representation of the braid with t strands. This turns out to be true [4]. Note that under this representation the braid is not necessarily a Lorenz braid.

Theorem 4.5. *Let L be the Lorenz link defined by the Lorenz vector $d_L = (d_1, \dots, d_p)$ and let $d_{\overline{L}} = (\overline{d}_1, \dots, \overline{d}_q)$ be the Lorenz vector of the dual Lorenz* *link $\overline{L}$. Let t be the trip number of L.*

Then L can be represented by a braid with t strands with the braid representation $XY'\Delta_t^2$ where $X = \prod_{i=1}^{p-t}[1, d_i - 1]$, $Y = (\prod_{i=1}^{q-t}[1, \overline{d}_i - 1])'$, and Y' is the braid word obtained from Y by replacing each σ_i with σ_{t-i}.

Proof. This proof is again similar to the proof of Theorem 4.1.

By untwisting and pushing the first $(p-t)$ strands to the top, the braid is transformed into

$$\begin{array}{c} X \\ [p, p+d_p-1] \\ [p-1, p+d_{p-1}-2] \\ \vdots \\ [p-t+1, p+d_{p-1}-2] \end{array}$$

, where X actually starts at the $(p+1)$-th strand, for convenience we still call it X.

The lower part is also a Lorenz braid L_1. By Lemma 4.2, X is a braid on at most t strands, hence it does not involve the last $(q-t)$ strands. As a result, the first $(q-t)$ digits of $d_{\overline{L_1}}$ is $\overline{d}_1, \ldots, \overline{d}_(q-t)$.

If we push the last $(q-t)$ up to the left, we obtain a braid of $2t$ strands with Y' on the top left, X on the top right and a Lorenz braid at the bottom. It is given by the vector (t^t), as the strands corresponds to the original LR and RL strands.

By sliding Y' down along the LR strands and then pushing the t strands on the left up. We get the required expression of the braid.

□

Corollary 4.1. *[4] A lorenz link is the trivial knot if and only if the trip number is 1.*

Next we turn to the question of how many strands are required to represent a Lorenz link.

Definition 4.3. *The **braid index** of a link L is the smallest number n such that L can be represented by a braid of n strands.*

It is known that the braid index is also equal to the minimal number $s(L)$ of Seifert circles.

In [4] it was conjectured that the trip number of a Lorenz link is the braid index. This is proved to be true in [11]. Here we follow the argument in [6].

Let $w(D)$ denote the writhe of a link diagram D. It is the sum of signs of crossings in D. Let $s(D)$ denote the number of seifert circles of D and let $\mu(L)$ denote the number of components of a link L.

Lemma 4.3. *Let D be an ascending diagram of a (trivial) link L. Then*

$$s(D) + w(D) \geq \mu(L).$$

Proof. The proof is by induction on $c(D)$. The result holds with equality when $c(D) = 0$. If $c(D) > 0$, it can be shown that there is a crossing such that smoothing it still gives an ascending diagram D'.

Now if it is a self crossing of a component, then

$$s(D) + w(D) = s(D') + w(D') \pm 1 \leq \mu(D') - 1 = \mu(D).$$

If it is a crossing between two components, then

$$s(D) + w(D) = s(D') + w(D') + 1 \leq \mu(D') + 1 = \mu(D).$$

□

Lemma 4.4. *Let L be an oriented link and L^* be the mirror image of L. Then*

$$P_{L^*}(v, z) = (-1)^{\mu(L)-1} P_L(v^{-1}, z).$$

Proof. The result follows immediately if we construct resolution trees of L and L^* with exactly the same crossings changed at each step and compare the polynomials along each path. □

The following result gives the lower bound of the braid index.

Theorem 4.6. *Let D be a diagram of a link L.*

Then

$$maxdeg_v P_L(v,z) \leq w(D) + s(D) - 1$$

$$mindeg_v P_L(v,z) \geq w(D) - s(D) + 1$$

Therefore,

$$\tfrac{1}{2} breadth_v P_L(v,z) + 1 \leq s(L).$$

Proof. Let $\phi(D) = w(D) + s(D) - 1$. To prove $maxdeg_v P_L(v,z) \leq \phi(D)$, we can prove $maxdeg_v[v^{-\phi(D)}P(D)] \leq 0$

By the skein relation,

$$\begin{aligned} v^{-\phi(D_+)}P(D_+) &= v^{-\phi(D_+)}v^2P(D_-) + v^{-\phi(D_+)}vzP(D_0) \\ &= v^{-\phi(D_-)}P(D_-) + v^{-\phi(D_0)}zP(D_0). \end{aligned}$$

Similarly, $$v^{-\phi(D_-)}P(D_-) = v^{-\phi(D_+)}P(D_+) - v^{-\phi(D_0)}zP(D_0).$$

Therefore, the result holds if it holds for each terminal node.

This is proved by an induction on $c(D)$. At the leftmost terminal node the diagram D' satisfies $maxdeg_v P_{D'}(v,z) = \mu(D') - 1$. By Lemma 4.3, $maxdeg_v P_{D'} \leq \phi(D')$. Hence we have the first inequality.

The second inequality holds because by Lemma 4.4,

$$\begin{aligned} mindeg_v P_L(v,z) &= -[maxdeg_v P_(L^*)v, z] \\ &\geq -[w(D^*) + s(D^*) - 1] \\ &= w(D) + s(D) + 1 \end{aligned}$$

□

Note that for a Lorenz braid of n strands and c crossings, $w(D) - s(D) + 1 = c - n + 1 > 0$ unless $c = n - 1$, in which case the braid has trip number 1 and corresponds to a trivial knot. Hence by the second inequality of Theorem 4.6 and Lemma 4.4, *any non-trivial Lorenz link is non-amphicheiral.* Actually, L. Rudolph proved in [19] that non-trivial positive braids have positive signature, which implies this result.

We continue the discussion on braid index. That the trip number t is the braid index follows from the fact that the representation in t strands is a positive braid with a full-twist.

Theorem 4.7. *Let $\beta \in \mathbb{B}_n$ be a positive braid. Let L be the closure of $\beta\Delta_n^2$. Then the braid index of L is equal to n.*

Proof. Let β ends with the generator σ_i. Note that the full-twist $\beta\Delta_n^2$ can be written in the form $\sigma_i\Delta'$ for any $i \in 1, \ldots, n$:

$$\begin{aligned}
\Delta_n^2 &= [1, n-1]^n \\
&= [1, n-1]^{i-1}[1, n-i][n-i+1, n-1][1, n-1]^{n-i} \\
&= [i, n-1][1, n-1]^{n-1}[1, i-1].
\end{aligned}$$

Smoothing the crossing corresponding to this σ_i gives

$$\begin{aligned} P(\beta\Delta_n^2) &= P(\beta'\sigma_i^2\Delta') \\ &= v^2P(\beta'\Delta') + vzP(\beta'\Delta_n^2) \end{aligned}$$

Repeat this process we can construct a resolution tree such that along the rightmost path we arrive at Δ_n^2. Continue the resolution by changing positive crossings only. Then $P(L)$ is of the form $\sum_j m_i(v,z)\delta^{\mu_j - 1}$, where each $m_i(v,z)$ is a product of positive powers of v and z.

Following the leftmost path from the node with Δ_n^2, there is a terminal node with n components. Hence $breadth_v P(L) \geq 2(n-2)$.

By Theorem 4.6,

$$n \geq s(L) \geq \tfrac{1}{2} breadth_v P_L(v,z) + 1 = n.$$

□

Chapter 5

Modular knots

In [8] E. Ghys proved that the isotopy classes of Lorenz knots and modular knots coincides. In this chapter we give a description of this correspondence. Most of the arguments are from [8], [7], [16].

5.1 The Modular flow

Given two complex numbers ω_1, ω_2 such that $\frac{\omega_2}{\omega_1} \notin \mathbb{R}$, they generate a lattice $\Lambda = \{m\omega_1 + n\omega_2 | m, n \in \mathbb{Z}\}$ in $\mathbb{C}$.

We focus on the space M of lattices of area one. Such a lattice is generated by a pair of complex numbers $\omega_2 = ai + b$, $\omega_1 = ci + d$, where $ad - bc = 1$.

Hence given a matrix $M \in PSL(2, \mathbb{R})$, it naturally determines a lattice of area one by $M \mapsto M \begin{bmatrix} i \\ 1 \end{bmatrix} (= \begin{bmatrix} \omega_2 \\ \omega_1 \end{bmatrix})$.

The lattice is invariant under a change of basis. This corresponds to multiplying an element of $PSL(2, \mathbb{Z})$ to $\begin{bmatrix} \omega_2 \\ \omega_1 \end{bmatrix}$. In $PSL(2, \mathbb{R})$ this corresponds to left multiplication by an element of $PSL(2, \mathbb{Z})$.

Hence we can identify M with $PSL(2, \mathbb{R})$.

On the other hand, given a lattice Λ, define

$$g_2(\Lambda) = 60 \sum_{\omega \in \Lambda \backslash \{0\}} \frac{1}{\omega^4} \text{ and } g_3(\Lambda) = 140 \sum_{\omega \in \Lambda \backslash \{0\}} \frac{1}{\omega^6}$$

It is well-known that the space of lattice can be identify with $\{(g_2, g_3) \in \mathbb{C}^2 | \Delta = g_2^3 - 27g_3^2 \neq 0\}$. Since $\Delta(\lambda\Lambda) = \lambda^{-12}\Delta(\Lambda)$, by rescaling, each lattice of area one has a unique representative on $\mathbb{S}^3 \cap \{\Delta \neq 0\}$. The set $K = \mathbb{S}^3 \cap \{\Delta = 0\}$ is a trefoil knot. This identifies M with the complement of a trefoil knot.

On M, a flow ϕ_t is defined as follows:

For a lattice Λ of area one generated by ω_1, ω_2. By considering ω_1, ω_2 as vectors in $\mathbb{R}^2$, $\phi_t(\Lambda)$ is defined as the lattice generated by $\begin{bmatrix} \omega_2 \\ \omega_1 \end{bmatrix} \begin{bmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{bmatrix}$.

Note that the resultant matrix has determinant 1. Hence the flow preserves the area of the lattice. As a change of basis corresponds to multiplication of a matrix on the left, the flow is well defined. This flow is called the *Modular flow.* The periodic orbtis of the Modular flow in $\mathbb{S}^3 \backslash K$ is called a *Modular knot.*

5.2 Modular Knots

The modular flow on M can actually be identified with the geodesic flow of the modular surface $\Sigma_{mod} = \mathbb{H}/PSL(2, \mathbb{Z})$. To see this, we first identify $PSL(2, \mathbb{R})$ with the unit tangent bundle $U\mathbb{H}$ of $\mathbb{H}$.

$U\mathbb{H}$ can be identified with $\mathbb{H} \times \mathbb{S}$. Given (v, θ), there is a unique $g \in PSL(2, \mathbb{R})$ that takes $(i, \frac{\pi}{2})$ to (v, θ). Here $\frac{\pi}{2}$ represents the unit tangent vector at i pointing upwards and tangent to the imaginary axis. (v, θ) is identified with g. The action of $PSL(2, \mathbb{R})$ on $U\mathbb{H}$ is then the left multiplication in $PSL(2, \mathbb{R})$.

The geodesic flow on $\mathbb{H}$ moves a tangent vector along the geodesic defined by this vector with unit speed. In $PSL(2,\mathbb{R})$ this corresponds to multiplying the matrix $a_t = \begin{bmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{bmatrix}$ on the right. The orbit ga_t projects to a geodesic through $g(i)$.

The quotient space $U\mathbb{H}/PSL(2,\mathbb{Z})$ can be identified with the unit tangent bundle $U\Sigma_{mod}$. The geodesic flow descends to the geodesic flow on Σ_{mod}. Now we have an identification of M and $U\Sigma_{mod}$ and we see that the modular flow corresponds to the geodesic flow on Σ_{mod} by the very same formula.

Now consider the periodic orbits of ϕ_t.

A point $g \in PSL(2,\mathbb{R})/PSL(2,\mathbb{Z})$ is periodic under ϕ_t if $ga_t = Hg$ for some $H \in PSL(2,\mathbb{Z})$ and for some $t > 0$. i.e. $a_t = g^{-1}Hg$. Since $tr(a_t) = (e^{\frac{t}{4}} - e^{-\frac{t}{2}})^2 + 2 \geq 2$ if $t > 0$, H is a hyperbolic element of $PSL(2,\mathbb{Z})$.

Moreover, for $A \in PSL(2,\mathbb{Z})$,

$$Aga_t = AHg = (AHA^{-1})(Ag).$$

Hence the periodic orbits coincides with the conjugacy classes of hyperbolic elements in $PSL(2,\mathbb{Z})$. The orbits, when projected to the modular surface, are the axes of hyperbolic elements in $PSL(2,\mathbb{Z})$.

In the following, we will see that the conjugacy classes of the hyperbolic elements can be coded similarly as the Lorenz words([7]), and we will describe how this is seen in the modular surface.

First note that the modular group $PSL(2,\mathbb{Z})$ is generated by the elements $a = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$ with the relation $a^2 = b^3 = 1$. Hence each element of $PSL(2,\mathbb{Z})$ can be written as a unique reduced word.

Lemma 5.1. *Every element $PSL(2,\mathbb{Z})$ not conjugate to a, b nor b^2 is conjugate in $PSL(2,\mathbb{Z})$ to an element of which the reduced word starts with b and ends with a.*

Proof. By cyclic permutation of the reduced word, which corresponds to conjugation, the reduced word can be arranged to be as described in the lemma. The exceptions are those which conjugate to a, b or b^2. □

Now let $X = ba = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $Y = b^2a = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Theorem 5.1. *Every hyperbolic element in $PSL(2,\mathbb{Z})$ is conjugate in $PSL(2,\mathbb{Z})$ to a product of X and Y which contains at least one X and one Y. Such a product is unique up to cyclic permutation.*

Proof. By Lemma 5.1, any element in $PSL(2,\mathbb{Z})$ is conjugate to a product of X and Y. Since $tr(X^n) = tr(Y^n) = 2$ for any n and conjugation preserves the trace, the product contains at least one X and one Y. It is easy to verify that a product involving at least one X and Y have trace greater than 2. □

Therefore, a periodic orbit of the modular flow is given a code in X and Y, similar to the case of a Lorenz knot.

Consider the Poincaré disc model of the hyperbolic plane.(Fig.5.1). The center corresponds to $e^{i\frac{\pi}{3}}$ in $\mathbb{H}$. The imaginary axis is taken to the other two sides by X and Y respectively. These two sides are labeled X and Y respectively. Consider the periodic orbits starting at the imaginary axes and pointing into the middle region. Given an hyperbolic element by conjugasion and translate along the flow it is possible to choose a starting point on the imaginary axis, so this include all periodic orbits.

For such a periodic orbit ga_t, determined by a hyperbolic element H, each time it hits the side X or Y, identify the point of intersection back to the imaginary axis(by X^{-1} or Y^{-1}, depending on the side it hits) and consider the orbit $X^{-1}ga_t$ (or $Y^{-1}ga_t$) starting at the imaginary axis. Since the orbit is periodic, after finitely many steps we returned to the starting point g. Suppose the sequence of

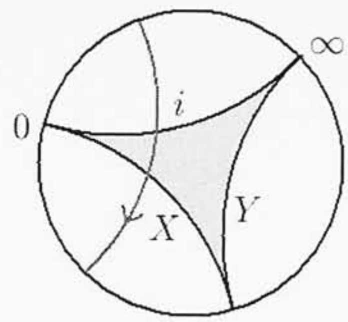

Figure 5.1:

sides being hit is $S_1, \ldots, S_n$, then $ga_{t'} = S_1 \ldots S_n g$, where t' is the period of ga_t. Hence H is represented by $S_1 \ldots S_n$. This gives an interpretation of the modular flow projected to the modular surface, which shows similarity with the Lorenz template.

Finally we state again the theorem of Ghys.

Theorem 5.2. *The isotopic classes of Lorenz knots and modular knots coincide. The same holds for Lorenz links and modular links.*

The idea of his proof is to deform the modular surface such that the cusp is opened. This results in an invariant region given by the convex hull of the limit set at infinity. The corresponding figure looks like Fig. 5.1, but the shaded part becomes a compact hexagon. He then shows that the periodic orbits can be collapsed onto a branched surface embedded in the complement of the trefoil knot, which corresponds to the Lorenz template. See [8], [9] for more details and animations for visualizing the orbits.

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